# THE TWENTY-SIXTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth summer meeting of the Society was held at the University of Michigan, Ann Arbor, Mich., September 2–4, 1919, in conjunction with meetings of the Mathematical Association of America and the American Astronomical Society. At a joint dinner of the three organizations on Thursday evening Mr. J. E. Beal, of the board of regents of the University, and Professor W. W. Beman, in the absence of the president and the dean, made addresses of welcome to the visiting societies, to which Professors Schlesinger, Eisenhart and Rietz gave appropriate responses in behalf of the societies; and a formal vote of thanks for the generous hospitality of the university was adopted and subsequently presented to the proper authorities.

The Michigan Union and Newberry Residence were opened for the accommodation of the visitors, meals being served in the former building. A complimentary luncheon given by the University on Thursday, an informal reception by Professor and Mrs. Hussey at the Detroit Observatory on Tuesday evening, and auto rides around Ann Arbor and to the neighboring city of Detroit were only three of the many kindnesses of the hosts. An exhibition of the astronomical and mathematical rarities in the University of Michigan Library attracted much attention.

A meeting of the Council of the Society was held Wednesday morning in Alumni Memorial Hall. Dr. Raymond W. Brink, of the University of Minnesota, was elected to membership in the Society, and six new applications for membership were received.

At the joint dinner on Thursday evening there were one hundred and eighty persons in attendance. At the regular scientific sessions of the Society there was an attendance of over a hundred, including the following eighty members of the Society:

Dr. E. S. Allen, Professor R. C. Archibald, Professor G. N. Armstrong, Professor W. W. Beman, Professor Susan R. Benedict, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Professor J. W. Bradshaw, Professor

W. C. Brenke, Professor E. W. Brown, Professor W. H. Butts, Professor W. D. Cairns, Professor A. L. Candy, Professor E. H. Clarke, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor S. C. Davisson, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor L. P. Eisenhart, Professor John Eiesland, Professor L. C. Emmons, Professor G. C. Evans, Professor Peter Field, Professor B. F. Finkel, Professor W. B. Ford, Mr. T. C. Fry, Dr. Elizabeth B. Grennan, Professor C. F. Gummer, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor H. A. Howe, Professor E. V. Huntington, Professor W. J. Hussey, Professor L. C. Karpinski, Professor A. J. Kempner, Professor A. M. Kenyon, Professor H. R. Kingston, Professor Florence P. Lewis, Professor G. H. Ling, Professor A. C. Lunn, Dr. E. B. Lytle, Professor J. V. McKelvey, Professor W. D. MacMillan, Professor J. L. Markley, Professor G. A. Miller, Professor E. J. Moulton, Professor F. R. Moulton, Dr. A. L. Nelson, Professor F. W. Owens, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor V. C. Poor, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Mrs. E. D. Roe, Professor W. H. Roever, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaught, Professor G. W. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor A. L. Underhill, Professor J. N. Van der Vries, Mr. H. E. Webb, Professor K. P. Williams, Professor B. F. Yanney, Professor J. W. Young, Professor Alexander Ziwet.

Thursday afternoon was devoted to a joint scientific session, at which Professor Slaught, President of the Mathematical Association, presided. The following addresses were given:

1. "Mathematics and statistics," by Professor E. V. HUNTINGTON, being the retiring address of the former president of the Mathematical Association.

2. "The work of the National Research Council with reference to mathematics and astronomy," by Professor E. W. BROWN.

3. "Report on the international conference of scientists at Brussels," by Professor FRANK SCHLESINGER, President-elect of the American Astronomical Society.

Professor Huntington gave an introductory account of the

mathematics of statistics and urged the mathematical fraternity to give the subject more consideration than it has received in the past. His address will appear in full in the *American Mathematical Monthly* in November.

Professor Brown commented on the lack of close cooperation, previous to the period of the war, amongst the various scientific societies of our country. About a year before our entry into the war the National Research Council was organized under the enthusiastic and capable leadership of Dr. G. E. Hale, partly as a preparedness measure, with the realization that in case of war cooperation would be imperative for efficient service from our scientific men. After our entry into the war innumerable problems of a scientific character were presented to the Council and through its aid were effectively treated. Now, in time of peace, it aims to encourage research and to aid in the promotion of science and scientific interests where a broad cooperation is desirable for best results. For much of its work the Council is divided, in the tentative plans, into a number of divisions, mathematics being associated with the physical sciences group, along with astronomy, physics and geophysics. Professor Brown emphasized the fact that the National Research Council did not wish in any way to interfere with the freedom of action of any of our scientific societies, or of individual scientists, but hoped primarily to aid in matters when cooperation was obviously desirable. Concrete illustrations of possible lines of activity were suggested, including the provision, with international cooperation, of bibliographies of science and the selection of a standard system of units and notation, the propaganda for the education of people to an appreciation of the importance of science, and the securing of government aid in the development of scientific projects.

Professor Schlesinger reported the organization of and results accomplished by the International Research Council which met at Brussels, July 18–30. American astronomers took an active part in the conference but no American mathematicians were present. Professor Schlesinger gave an account of the activities of the party of American astronomers and the organization of an International Astronomical Union. An extended report of the paper will be published in *Popular Astronomy*. Dr. L. A. Bauer, of the Carnegie Institution, who also attended the Brussels conference, spoke briefly of

the organization of an International Geophysical Union as well as the International Astronomical Union.

At the regular sessions of the Society for the presentation of research papers, held Tuesday afternoon, Wednesday morning and afternoon and Thursday morning, Professors Beman, Curtiss, Snyder and Bliss served as presiding officers. Professor Karpinski served as secretary at the first session, and Professor E. J. Moulton at the other sessions. The following thirty-three papers were read:

(1) Professor PETER FIELD: "On wind corrections."

(2) Professor H. J. ETTLINGER: "Cauchy's memoir of 1814 on definite integrals."

(3) Mr. L. H. RICE: "Expansion of any determinant in minors from rectangular panels."

(4) Dr. A. L. NELSON: "Pseudo-canonical forms and invariants of systems of partial differential equations."

(5) Professor A. J. KEMPNER: "On the separation of complex roots of an algebraic equation."

(6) Dr. C. N. REYNOLDS, JR.: "Some theorems on the zeros of solutions of homogeneous linear differential equations of the $n$th order."

(7) Dr. REYNOLDS: "Some theorems on the zeros of solutions of self-adjoint homogeneous linear differential equations of the fifth order."

(8) Dr. J. W. ALEXANDER: "Proof of the existence of distinct three-dimensional manifolds with the same group."

(9) Professor E. D. ROE, JR.: "Certain determinants expressible as circulants or skew-circulants."

(10) Professor W. B. FORD: "A brief account of the life and work of the late Professor Ulisse Dini."

(11) Dr. S. P. SHUGERT: "The resolvents of König and other types of symmetric functions."

(12) Professor G. A. MILLER: "Form of the number of subgroups of prime-power groups."

(13) Dr. E. S. ALLEN: "A generalization of a formula of Schubert in enumerative geometry."

(14) Dr. E. P. LANE: "Joint-axis congruences with indeterminate developables."

(15) Dr. R. W. BRINK: "A modification of an integral test for the convergence and divergence of infinite series."

(16) Professors F. R. SHARPE and VIRGIL SNYDER: "Certain types of involutorial space transformations (second paper)."

(17) Professor L. P. EISENHART: "Transformations of surfaces applicable to a quadric."

(18) Professor EISENHART: "Transformations of cyclic systems of circles."

(19) Professor G. A. BLISS: "Differential equations containing arbitrary functions."

(20) Professor BLISS: "Functions of lines in ballistics."

(21) Professor D. R. CURTISS: "On the relative positions of the complex roots of an algebraic equation with real coefficients and those of its derived equation."

(22) Professor H. L. RIETZ: "Urn schemata as a basis for the development of the theory of correlation."

(23) Professor L. L. DINES: "Projective transformations in function space (second paper)."

(24) Professor J. B. SHAW: "On the invariants belonging to a hypernumber in an algebra of infinite order."

(25) Professor R. D. CARMICHAEL: "Conditions necessary and sufficient for the existence of a Stieltjes integral."

(26) Professor CARMICHAEL: "Note on convergence tests for series and on Stieltjes integration by parts."

(27) Professor CARMICHAEL: "Note on a physical interpretation of Stieltjes integrals."

(28) Professor H. A. HOWE: "An apparent anomaly in errors of interpolated values."

(29) Professor G. C. EVANS: "Transformations of a Stieltjes integral potential."

(30) Professor T. H. HILDEBRANDT: "Note on sequences of Stieltjes integrals."

(31) Professor W. H. ROEVER: "Equations of motion of a projectile regarded as a particle."

(32) Professor W. D. CAIRNS: "Certain properties of binomial coefficients."

(33) Professor A. J. KEMPNER: "On the shape of polynomial curves."

In the absence of the authors the papers by Professor Ettlinger, Mr. Rice, Dr. Reynolds, Dr.. Alexander, Dr. Shugert, Dr. Lane, Dr. Brink, and Professor Carmichael, were read by title only. The papers on ballistics led to a somewhat extended discussion by Professors Field, Bliss, Roever, and F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles above:

1. Professor Field's paper deals with the fundamental assumptions on which the ordinary wind correction formulas are based. It will be published in the *Journal of the U. S. Artillery*.

2. In this paper Professor Ettlinger discusses Cauchy's memoir of 1814 from a historical and critical point of view. Eminent mathematicians of that day, such as Poisson, Lacroix and Legendre, did not properly appreciate the importance of the contribution and merely saw in the memoir new methods of evaluating definite integrals which Euler had computed years before. For the modern reader, also, it is difficult to find in this memoir the very first statement of Cauchy's discoveries on residues and the integral theorem upon which the structure of the modern theory of functions of a complex variable is built. These facts the author brings into the light by simplifications, both algebraic and geometrical. The method is also made to conform to our modern standard of rigor.

The subject of this paper was suggested by Professor W. F. Osgood.

3. By availing himself of the full force of a new proof of Muir's recent theorem (*Messenger of Mathematics*, May, 1918) that "a determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another," Mr. Rice shows, not only that the last clause of the theorem can be freed of the restriction to coaxial (square) arrays, but also that there need not be precisely four rectangular arrays or panels but the matrix of the determinant may be subdivided into any number of panels independent of one another in point of dimensions, complementary sets of minors being drawn from these panels.

4. In a recent note (*American Journal of Mathematics*, volume 41, April, 1919, pages 123–132), Dr. Nelson has shown that fundamental sets of seminvariants of a completely integrable system of linear homogeneous partial differential equations may be found by the reduction of the system to certain pseudo-canonical forms. The method is here extended to include semi-covariants, invariants and covariants. Simul-

taneous invariants of several such completely integrable systems may be similarly constructed. Application is also made to semi-covariants of algebraic forms.

5. By simple vector considerations Professor Kempner determines certain sectors starting at the origin in the complex plane which do not contain roots of a given algebraic equation. These sectors depend, in case the equation has real coefficients, only on the sequence of signs of the coefficients; in case the equation has complex coefficients, only on the arguments of the coefficients. When the equation is of low degree or when, while the degree of the equation is high, many coefficients vanish, the information obtained is often sufficient to separate entirely the complex roots. In most cases the information is less precise. The method applies also to equations containing fractional powers of the unknown.

6. In this paper Dr. Reynolds proves a general separation theorem for the zeros of solutions of homogeneous linear differential equations of the $n$th order and compares the zeros of the solutions of the two differential systems

$$(1)\qquad \begin{aligned} &y^{(n)}(x) + \sum_{i=2}^{n} p_i(x) y^{(n-i)}(x) = 0, \\ &y^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \cdots, n-1), \end{aligned}$$

$$(2)\qquad \begin{aligned} &\eta^{(n)}(x) + \sum_{i=2}^{n-1} p_i(x) \eta^{(n-i)}(x) + [p_n(x) + R(x)]\eta(x) = 0, \\ &\eta^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \cdots, n-1). \end{aligned}$$

For $n$ even the theorems obtained generalize the author's earlier results concerning equations of the fourth order. For $n$ odd the theorems obtained generalize Professor Birkhoff's results concerning equations of the third order (*Annals of Mathematics*, second series, volume 12).

7. Dr. Reynolds applies the general separation theorem of the preceding paper to the solutions of self-adjoint homogeneous linear differential equations of the fifth order. Any solution of such an equation can be expressed linearly in terms of the six functions $\eta_i\eta_j' - \eta_j\eta_i'$ $(i, j = 1, 2, 3, 4; i < j)$, where the $\eta$'s satisfy a homogeneous self-adjoint linear differential

equation of the fourth order. The wronskian of any two solutions of the fifth order equation is a homogeneous quadratic form in the $\eta$'s. By means of these two relations the author proves a group of theorems concerning the solutions of self-adjoint equations of the fifth order which are of the same degree of generality as his earlier results concerning self-adjoint equations of the fourth order and interprets his results geometrically by means of the integral curve of the self-adjoint equation of the fourth order which is satisfied by the $\eta$'s.

8. In this paper, Dr. Alexander shows that two distinct three-dimensional manifolds with the same Betti numbers, coefficients of torsion, and group exist. The Heegaard diagrams of the manifolds consist of anchor rings on which have been traced the curves $ab^5$ and $a^2b^5$ respectively.

9. Generalizing, Noether and Puchta discovered determinants of $m$ elements, and of order $m$, which were factorable into $m$ linear factors of their elements, the factors being real for $m = 2^n$. Noether showed that, for $m = n_1 n_2 \cdots n_k$, the determinant is the simultaneous resultant of a first set of equations $x_1^{n_1} - 1 = 0$, $x_2^{n_2} - 1 = 0$, $\cdots$, $x_k^{n_k} - 1 = 0$, and $\Sigma a_{i_1 i_2 \cdots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} = 0$, where the degree of $\Sigma$ in $x_1$, $x_2$, $\cdots$, $x_k$ is $n_1 - 1$, $n_2 - 1$, $\cdots$, $n_k - 1$ respectively. Professor Roe extends the generalization to the resultant of a second set of equations $x_1^{n_1} + 1 = 0$, $x_2^{n_2} + 1 = 0$, $\cdots$, $x_k^{n_k} + 1 = 0$, and $\Sigma = 0$. He points out that by eliminating one variable $x_j$ by Sylvester's method a circulant or a skew-circulant respectively, in the remaining variables, of order $n_j$ results. The elimination of a second variable, not by Sylvester's method, but by forming the product of the function obtained by substituting all the values of that variable, obtained from its equation in the first or second set respectively, in the circulant or skew-circulant already formed, yields, by the laws for the multiplication of circulants or skew-circulants, a circulant or skew-circulant of order $n_j$, and containing one less variable. Continuation of the elimination in this way, step by step, gives the resultant finally as a circulant or skew-circulant of order $n_j$. As the variable $x_j$ first eliminated was chosen at pleasure, it follows that the determinants of Noether and Puchta and their analogues

here suggested by Professor Roe can be expressed as circulants or skew-circulants respectively of any of the orders $n_1$, $n_2$, $\cdots$, $n_k$. Numerous identities follow from these different expressions. As a simple example of such identities, let $k = 2$, $n_1 = 2$, $n_2 = 3$. Then from one or the other form we can derive $A^3 + B^3 \equiv a^2 - b^2$, $A'^3 - B'^3 \equiv a'^2 + b'^2$, where $A$, $B$, $A'$, $B'$, $a$, $b$, $a'$, $b'$ are functions of 5 independent of 6 elements, another function $C$ or $C'$ being placed equal to zero to secure the form. The identities are satisfied in $\infty^5$ ways. They can be generalized to such as are satisfied in $\infty^{2n-1} = \infty^{3p-1}$ ways by $2n - 1$ independent elements, where $n \equiv 0 \pmod 3$, $p \equiv 0 \pmod 2$, and these in turn can be still further generalized. Professor Roe also considers the resultant of the elimination between a third set of equations $x_1^{n_1} - 1 = 0, \cdots, x^{n_k} - 1 = 0$, $x^{m_1} + 1 = 0, \cdots, x^{m_e} + 1 = 0$ and $\Sigma = 0$, by virtue of which the resultant can be shown to be a circulant of any of the orders $n_1, \cdots, n_k$ or a skew-circulant of any of the orders $m_1, \cdots, m_e$. He also considers the resultant between any of the systems and $\Sigma = 0$, when all restrictions as to the degree of $\Sigma$ are removed, and shows that according as $d_j \gtrless n_j - 1$ ($d_j$ the degree of $x_j$ in $\Sigma$), circulants or skew-circulants with polynomial or zero elements result. Also such identities as $A^2 + B^2 \equiv A'^2 - B'^2$, $A^3 + B^3 \equiv A'^3 - B'^3$, etc.

10. Professor Ford's paper is intended to supplement the brief announcement of Professor Dini's death as recorded in the BULLETIN of last March and April.

11. Dr. Shugert's paper treats the following subjects, in part by methods and formulas derived or suggested by Professor Glenn in a paper on the theory of finite expansions in volume 15 of the *Transactions*: a lemma on determinants whose elements above the diagonal next above the principal diagonal are all zero; the expansion of a polynomial in ascending powers of a quadratic polynomial; conditions that a form contain a given quadratic form as a factor; separation of $a_x^m/(\xi_x^2)^k$ into partial fractions; the explicit formation, by determinants, of the resolvents studied by König in the *Mathematische Annalen*, volume 15 (1879); tables of symmetric functions of the sums and of the products, in pairs, of the roots of a given binary form. These tables are complete for the forms of orders 2 to 6 inclusive.

The paper is a University of Pennsylvania doctor dissertation and is being printed by the New Era Press by private arrangement with the author.

12. Professor Miller's paper appears in full in the present number of the BULLETIN.

13. If, on an algebraic curve of genus $p$, an algebraic series of point groups $\gamma_m^1$ of dimension 1 and a linear series $g_n^r$ of any dimension $r$ are given, the formula of Schubert states the number of those groups of $\gamma_m^1$ which contain $r+1$ points of a single group of $g_n^r$.

Dr. Allen obtains the number of those groups of a series $\gamma_m^2$, of dimension 2, which contain $r+2$ points of a single group of a given $g_n^r$. He obtains a recurrence formula by means of two pairs of correspondences between points of the curve, and from this finds the closed formula by mathematical induction.

The formula has the following two simple forms, which involve, respectively, the characters $z_b$ of the series $\gamma_m^2$, and the numbers $d_{1[a]}$, of groups of $\gamma_m^2$ possessing $a$ double points, together with enough arbitrary fixed points to make $d_{1[a]}$ result finite:

$$Z_{n,\,r;\,m,\,2} = \sum_{b=0}^{2}\sum_{h=b}^{2}(-1)^{h+b}\binom{m-2-h}{r-h}\binom{n-r-h}{2-h}\binom{p-b}{h-b}z_b$$

$$= \sum_{a=0}^{2}\sum_{h=a}^{2}(-2)^{-a}\binom{m-2-a}{r-a}\binom{r-a}{h-a}\binom{n-r-h}{2-h}d_{1[a]}.$$

14. The joint-axis of a point $P_y$ on a surface $S_y$, referred to a conjugate net, and of the corresponding point $P_z$ on the first Laplace transform $S_z$ of the net, has been defined by Wilczynski to be the line of intersection of the osculating plane of the curve $u = \text{const.}$ at $P_y$ and the osculating plane of the curve $v = \text{const.}$ at $P_z$. Thus, all the joint-axes of pairs of corresponding points constitute a congruence. Dr. Lane studies the properties of the joint-axis congruence, and of the suite of Laplace transforms of $S_y$, which are characteristic of this configuration when the developables of the joint-axis congruence are indeterminate. He finds, by the methods of projective differential geometry, using the nota-

tion of Wilczynski's system ($D$), that there are two types of such configurations. In the first type, the joint-axes all pass through a fixed point, to which the third and minus second Laplace transforms of $S_y$ reduce. In the second type, the joint-axes all lie in a fixed plane and give rise to a planar net of period three. The plane is characterized geometrically, and a number of geometrical theorems are obtained which relate certain points and lines in the configuration which have escaped notice hitherto.

15. In a previous paper Dr. Brink presented an integral test for the convergence and divergence of infinite series. It is applicable to series for which a simple analytic expression for a function $r(x)$ is known, where $r(n)$ is the ratio of the $n$th term of the series to the preceding term. And in this test $r(x)$ plays a rôle analogous to that played in the familiar Maclaurin-Cauchy integral test by the function $u(x)$, where $u(n)$ is the $n$th term of the series. In the present paper is given a modification of this test, in which a new form of the integral is introduced. This permits of a certain simplification of the conditions of the theorem, and a great simplification in the demonstration.

16. The paper of Professors Sharpe and Snyder is supplementary to that having the same title in the July *Transactions*. A general method of determining (1, 2) correspondences is developed and is applied to discuss the possible cases in which the order of the surfaces that are mapped on the planes of the double space does not exceed five. The discussion includes among others those involutions in which the surface of invariant points is birationally equivalent to a quartic with ten or more nodes.

17. In volume 19 of the *Transactions* Professor Eisenhart established a transformation of a pair of applicable surfaces into new pairs of applicable surfaces. In the present paper he shows that the transformations of surfaces applicable to a central quadric determined at great length by Guichard in the memoir which received the grand prize of the French academy in 1909 are of this kind, and that they follow readily from the general theory. Similar transformations of surfaces applicable to a paraboloid are established. The known trans-

formations of surfaces of constant curvature are likewise of this type. The general transformations of surfaces applicable to the quadrics admit two theorems of permutability, in consequence of which from known transformations others follow directly.

18. When the circles of a two-parameter family of circles are orthogonal to a one-parameter family of surfaces $\Sigma$, the circles are said to form a cyclic system. In the correspondence established between the surfaces by the circles, the lines of curvature correspond on all the surfaces, and to them correspond also the developables of the congruence formed by the axes of the circles. The planes of the circles envelope a surface $S$ upon which the curves corresponding to the lines of curvature on the surfaces $\Sigma$ form a conjugate net $N$, which admits an applicable conjugate net $\overline{N}$. Professor Eisenhart applies to these applicable nets the theorem referred to in his former paper and establishes thereby a transformation of the given cyclic system into other cyclic systems. Corresponding circles of a cyclic system and a transform lie on a sphere, and the nets $N$ and $N_1$ on the surfaces $S$ and $S_1$ enveloped by the planes of the circles are in relation $T$, that is, the developables of the congruence of joins of corresponding points on $S$ and $S_1$ meet the latter in $N$ and $N_1$. Moreover, it is possible to establish a correspondence between the surfaces $\Sigma$ and $\Sigma_1$, orthogonal to the two families of circles, so that a surface $\Sigma$ and the corresponding surface $\Sigma_1$ are in the relation of a transformation of Ribaucour, that is, they are the envelope of a two-parameter family of spheres with lines of curvature in correspondence on the two sheets of the envelope.

19. If in a set of differential equations

$$\frac{dx_i}{d\tau} = f_i(\tau, x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

the functions in the second members are considered as variables, the solutions $x_i$ $(i = 1, \cdots, n)$ will be functions, not only of the initial values of $\tau$ and the $x$'s, but also of the functions $f_i$ themselves. In Professor Bliss's first paper the properties of these functions of functions are discussed. It is proved, after suitable initial hypotheses have been made, that they are continuous, and that they have what are called

in this paper difference functions. It is shown that the possession of a difference function implies also the possession of a differential similar to those which have been defined by Fréchet and Volterra for functions of lines. The theorems of the paper are widely applicable since the variations $\Delta f_i$ of the functions $f_i$ entering into the differentials are entirely arbitrary. In special cases they may be caused, for example, by variations of parameters in the functions $f_i$, or, as in ballistics, by variations of other arbitrary functions which occur as arguments in the second members of the differential equations.

20. In Professor Bliss's second paper some functions of lines are considered which arise in ballistics. If $w(y)$ is the velocity of the wind at the altitude $y$ in the direction of the plane of fire of a projectile, then the range $X$ will be a function of the function $w(y)$, and also of the projections $x_0'$, $y_0'$ on the axes of the initial velocity of the projectile. The first differential of this function $X[x_0', y_0', w(y)]$ is the first order effect on the range of wind and of changes in the direction and magnitude of the initial velocity. The paper gives a method of computing the first differential which is of interest theoretically in the theory of functions of lines, and which has proved to be of value in the practical computation of range tables. The situation just described is somewhat simpler than that which arises in practice, for the range is also affected by variations from normal in the density of the air and by the rotation of the earth, and still other disturbing influences will have to be accounted for in the future as the applications of ballistic theory become more exacting. The formulas of the paper are applicable to the cases which have just been mentioned explicitly, and they will undoubtedly also be useful for the computation of other differential corrections which may be necessary in the future. The results obtained are special cases of those of the preceding paper, but they are here deduced independently except for some general theorems which would very likely be accepted by an applied mathematician as intuitively true.

21. Jensen has recently stated, without proof, a theorem regarding the roots of the equations $f'(z) = 0$ and those of $f(z) = 0$, where the coefficients are all real, which locates the

former in many cases more closely than does the Gauss-Lucas "polygon theorem." Professor Curtiss's paper gives a simple proof of Jensen's theorem, and deduces from it the result that at least one pair of roots of $f(z) = 0$ lies in each rectangular hyperbola whose vertices are a conjugate imaginary pair of roots of $f'(z) = 0$. This result is extended so as to give similar theorems regarding the relative positions of roots of $f(z) = 0$ and its derived equations of all orders.

22. It is well known that simplicity and precision are gained by the use of urn schemata in establishing various theorems in the theory of probability. The fundamental importance of urn schemata in mathematical statistics is brought out well by Borel in his Eléments de la Théorie des Probabilités by the statement that the general problem of mathematical statistics is to determine a system of drawings carried out with urns of fixed composition, in such a way that the results of the series of drawings, interpreted by the use of conveniently chosen coefficients, lead, with a very high degree of probability, to a table of values identical with the table of observed values. In the present paper, Professor Rietz presents several urn schemata to show the significance of the correlation coefficient derived from a priori probabilities or most probable frequencies given by pure games of chance. It turns out that in certain cases the correlation coefficient $r$ is simply the ratio of the number of common elements to the total number involved. In other cases, the results, though not so simple, are still capable of interesting interpretations. It is emphasized that such urn schemata may well be made the starting point in the development of the theory of correlation.

23. The theory of the transformations studied by Professor Dines in a paper of the same title in the *Transactions* (volume 20, January, 1919) is further developed in the present paper. The introduction of homogeneous coordinates is attended by the usual advantages. A class of points satisfying a certain type of linear equation is called a lineoid, and serves as an element dual to the point. The question of invariant elements is treated, and provides a means for classifying projective transformations. A theorem of special interest states: A necessary and sufficient condition that a projective trans-

formation be projectively equivalent to an ordinary Fredholm transformation in the same domain is that it admit an invariant lineoid and an invariant point not on that lineoid. In particular every symmetrical projective transformation has this property, but such is not the case with non-symmetrical transformations.

24. The invariants referred to in Professor Shaw's paper are the scalar invariants which appear in the characteristic equation, the expressions previously called *chi*, and related forms. The subject is intimately connected with the Fredholm theory in integral equations.

25. The purpose of the first paper by Professor Carmichael is to suggest a method for deriving a necessary and sufficient condition for the existence of the Stieltjes integral of $f(x)$ as to $u(x)$ from $a$ to $b$ in each of several forms generalizing those frequently employed in the special case of Cauchy-Riemann integration. The theorem of Bliss (*Proceedings of the National Academy of Sciences*, volume 3 (1917), pages 633–637) is included and is proved in a new way.

26. An identity to which he was led by the problem of Stieltjes integration by parts Professor Carmichael applies in the derivation of several theorems relating to the convergence of series. Of these the following is typical: The convergence of the series

$$\sum_{k=1}^{\infty} c_{k-1}{}^{(2)} c_{k-1}{}^{(3)} \cdots c_{k-1}{}^{(n)} (c_k{}^{(1)} - c_{k-1}{}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} |c_k{}^{(i)} - c_{k-1}{}^{(i)}|, \quad (i = 2, 3, \cdots, n),$$

and the existence of the limits

$$\lim_{k=\infty} c_k{}^{(1)} c_k{}^{(2)} \cdots c_k{}^{(n-1)}, \quad \lim_{k=\infty} c_k{}^{(1)} \cdots c_k{}^{(i-1)} c_{k-1}{}^{(i+1)} \cdots c_{k-1}{}^{(n)}, \quad (i = 2, \cdots, n-1).$$

The instance $n = 2$ of this theorem is classic.

27. In this paper Professor Carmichael throws Stieltjes'

problem of moments into a different form from that of Stieltjes and one readily capable of generalization; and from this he proceeds to a natural physical interpretation of the Stieltjes integral of a bounded function $f(x)$ as to a function $v(x)$ of bounded variation.

28. On page 506 of Merriman and Woodward's Higher Mathematics is given a table showing—for different values of the interpolating factor—the actual and the theoretical average errors of certain interpolated values. Attention is called to the curious fact that when the interpolating factor is an even number the actual error is always less than the theoretical error, while when it is odd the reverse is true, except when the factor is 0.5.

In Professor Howe's paper it is purposed to describe a more exhaustive test of this matter, with special attention to the case where the interpolating factor is 0.5, and also to call attention to an Austrian five-place table from which more accurate results can be obtained than from ordinary tables. Also when only the usual accuracy is sought one can work much more rapidly with this table than with others.

29. In the current volumes of the *Rendiconti* of the Lincei and of the *Seminario* of the University of Rome, Professor Evans expounded a theory of the general equation of Poisson,

$$\int \frac{\partial u}{\partial n} ds = f(e),$$

where $f(e)$ is an arbitrary additive function of point sets, in which he introduced a potential function in the form of a Stieltjes integral. The present report discusses a transformation of this integral into the form involving a polarization vector, and a second form obtained by the slight generalization of an integration by parts of W. H. Young. In the latter case are introduced nuclei which in three dimensions attract according to an inverse fifth power law.

30. In this note Professor Hildebrandt derives a necessary and sufficient condition in order that a sequence of Stieltjes integrals $\int f d\alpha_n$ shall converge to an integral $\int f d\alpha$, for every continuous function on the linear interval $a \leqq x \leqq b$, the functions $\alpha_n$ and $\alpha$ being of bounded variation on the same

interval. He also shows that a linear functional operation on the class of functions of bounded variation on the interval $(a, b)$ is expressible in the form of a Stieltjes integral $\int f d\alpha$, in which $f$ is a continuous function on $(a, b)$; and derives necessary and sufficient conditions that the sequence of Stieltjes integrals $\int f_n d\alpha$ shall converge to an integral of the form $\int f d\alpha$, for every function of bounded variation on the interval $(a, b)$, the functions $f_n$ and $f$ being continuous on the same interval.

31. Professor Roever's paper begins with a statement of the theorems of relative motion concerning the velocities and accelerations of a particle. By applying these theorems to the problem under consideration the equation of motion of a projectile is expressed in vector form in terms of the accelerations due to weight, wind, air resistance and rotation of the earth. From this vector equation the equations of motion are obtained in a form involving the potential function $W$ of the weight field of force, the function $F$ which expresses the retardation due to air resistance and wind, and the angular velocity of the earth's rotation. By replacing $W$ by means of some of its analytic approximations and expressing $F$ in terms of various experimentally determined functions, differential equations of motion of a projectile are obtained which take into consideration air resistance, wind, rotation of the earth, convergence of the verticals, curvature of the layers of constant air density and also other influences, but not the rotation of the projectile due to rifling of the gun.

32. There are known summations of products of the successive coefficients of $(x - y)^k$ by given powers of the numbers of those coefficients, but no results seem to be known for the case of the binomial coefficients, i.e., those of $(x + y)^k$. A recursion formula is given by Professor Cairns for such summations as well as expressions for the leading terms, and an application is made to the evaluation of

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^{2p} dx \quad \text{and} \quad \int_{0}^{\infty} e^{-x^2/2\sigma^2} x^{2p+1} dx.$$

33. Professor Kempner gives a simple method of classification of real polynomial curves with reference to the relative

position and the number of the maxima and minima of the curve, and shows that all types actually exist. For example: for all $\delta > 0$ sufficiently small, the curve

$$y = (x + \delta i)(x - \delta i)(x + 1 + \delta^3 i)(x + 1 - \delta^3 i) \cdot$$
$$(x + 1 + \delta^2 + \delta^5 i)(x + 1 + \delta^2 - \delta^5 i)(x + 1 + \delta^2 + \delta^4)$$

has six extremes which, read for decreasing values of $x$, are arranged so that the first minimum of $y$ is higher than the second maximum, and the second minimum higher than the third maximum.

E. J. MOULTON,
*Acting Secretary.*

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## FORM OF THE NUMBER OF SUBGROUPS OF PRIME POWER GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 3, 1919.)

### §1. *Introduction.*

IT is known that the number of the subgroups of order $p^\alpha$, $p$ being any prime number, which are contained in any group $G$ is always of the form $1 + kp$. When $k = 0$ for every possible pair of values for $\alpha$ and $p$ the group $G$ must be cyclic and vice versa. There are two infinite systems of groups of order $p^m$ containing separately $p + 1$ subgroups of every order which is a proper divisor of the order of the group, viz., the abelian groups of type $(m - 1, 1)$ and the conformal non-abelian groups.

These two infinite systems are composed of all the groups of order $p^m$ involving separately exactly $p + 1$ subgroups of every order which is a proper divisor of $p^m$. Moreover, if a group of order $p^m$, $p > 2$, contains exactly $p + 1$ subgroups of each of the two orders $p$ and $p^2$ it must contain exactly $p + 1$ subgroups of every order which is a proper divisor of the order of the group, and if a group of order $2^m$ contains exactly three subgroups of each of the orders 2, 4 and 8 it must also contain exactly three subgroups of every other order which is a proper divisor of $2^m$.

The proof of the former part of this theorem may be based upon the fact that when $m > 3$ such a group has to contain an invariant abelian subgroup of order $p^3$ and of type (2, 1). The order of each of the remaining operators must be divisible by $p^3$ and the number of the operators of order $p^3$ is equal to $p^4 - p^3$. Hence $G$ contains exactly $p$ cyclic subgroups of order $p^3$ and only one non-cyclic subgroup of this order. If $m > 4$, this subgroup of order $p^4$ must again be abelian, and by successive similar steps we note that $G$ must contain operators of order $p^{m-1}$.

When $p = 2$ it is possible to extend the abelian group of order 8 and of type (2, 1) by 24 operators of order 4 so as to obtain a group of order 32. In this case the preceding reasoning fails. If it is assumed, however, that $G$ contains exactly three subgroups of order 8 in addition to the three subgroups of each of the orders 2 and 4 it may be proved as in the preceding paragraph that $G$ contains only $p = 2$ cyclic subgroups of order 16, etc. The theorem stated in the second paragraph has therefore been established. This theorem may be compared with the well known theorem, due to W. Burnside, which affirms that every group of order $p^m$, $p > 2$, which contains only one subgroup whose order is a proper divisor of $p^m$ is cyclic.

Another simple condition which implies that a group $G$ of order $p^m$ is cyclic is that all the operators of $G$ which transform into itself one of its subgroups constitute a cyclic subgroup of $G$. In other words, a necessary and sufficient condition that a prime power group is cyclic is that it contains at least one subgroup whose normaliser is cyclic. This theorem results directly from the fact that every non-invariant subgroup of a group of order $p^m$ is transformed into itself by at least $p$ of its conjugates including itself. In the following section it will be proved that a necessary and sufficient condition that an abelian group of order $p^m$ is cyclic is that the number of its subgroups of order $p^\alpha$, $0 < \alpha < m$, is of the form $1 + kp^2$ for at least one value of $\alpha$.

## §2. *Abelian Groups.*

Let $H$ represent a subgroup of order $p^\alpha$ contained in an abelian group $G$ of order $p^m$ and suppose that $H$ has been so selected that its invariants are as small as possible when the

order of $H$ is fixed. It may happen that $H$ is composed of all the operators of $G$ whose orders do not exceed the largest invariant of $H$. If this is the case, $G$ contains only one subgroup of order $p^\alpha$ having the same invariants as $H$. If it is not the case, we proceed to prove that the number of subgroups of $G$ which have the same invariants as $H$ must be of the form $1 + p + kp^2$.

Let $p^{\alpha+\lambda}$ represent the order of the subgroup of $G$ composed of all its operators whose orders divide the largest invariant of $H$. If $H$ has $\beta$ such largest invariants, the $\beta$ independent generators of $H$ whose orders are equal to these invariants can be selected from the operators of $G$ in a number of ways represented by the following product:

$$(p^{\alpha+\lambda} - p^{\alpha-\beta})(p^{\alpha+\lambda} - p^{\alpha-\beta+1}) \cdots (p^{\alpha+\lambda} - p^{\alpha-1}).$$

Similarly, these $\beta$ independent generators can be selected from the operators of $H$ in

$$(p^\alpha - p^{\alpha-\beta})(p^\alpha - p^{\alpha-\beta+1}) \cdots (p^\alpha - p^{\alpha-1})$$

different ways, and the remaining independent generators can be selected in the same number of ways from the operators of $G$ and those of $H$. When $\lambda > 0$, the quotient of the first of these two products divided by the second is evidently of the form $1 + p + kp^2$, and hence the following theorem has been established:

THEOREM. *The number of the subgroups of order $p^\alpha$ contained in an abelian group of order $p^m$ and satisfying the condition that their invariants are as small as possible when $\alpha$ is given is either unity or of the form $1 + p$, mod $p^2$.*

As a special case of this theorem it may be noted that the number of subgroups of order $p^\alpha$ contained in an abelian group of order $p^m$, $m > \alpha > 0$, and of type $(1, 1, 1, \cdots)$ is always of the form $1 + p$, mod $p^2$. In this special case $\alpha$ and $\beta$ are always equal to each other.

Suppose that $G$ contains only one subgroup $H$ of order $p^\alpha$ such that its invariants are as small as possible and consider the subgroups of order $p^\alpha$ which have the property that their largest invariant is $p$ times a largest invariant of $H$ and that a second invariant is equal to one of the largest remaining invariants of $H$ divided by $p$ while the rest of the invariants of such a subgroup $H'$ are the same as those of $H$. We shall

prove that the number of the subgroups of $G$ satisfying the conditions imposed on $H'$ is always of the form $p + kp^2$, so that in this case the number of subgroups of $G$ which satisfy the conditions imposed on $H$ and $H'$ is $1 + p \bmod p^2$.

It will first be assumed that $H$ contains $\beta > 2$ largest invariants and that the subgroup of $G$ composed of all its operators whose orders divide this largest invariant multiplied by $p$ is of order $p^{\alpha+\lambda}$. Hence $\lambda \leqq \beta \leqq \alpha$. The number of ways in which a set of $\beta - 1$ largest independent generators of $H'$ can be selected from the operators of $G$ is expressed by the following product:

$$(p^{\alpha+\lambda} - p^{\alpha})(p^{\alpha} - p^{\alpha-\beta+1}) \cdots (p^{\alpha} - p^{\alpha-2}).$$

The number of ways in which these generators can be selected from the operators of $H'$ is represented by the following product:

$$(p^{\alpha} - p^{\alpha-1})(p^{\alpha-1} - p^{\alpha-\beta+1}) \cdots (p^{\alpha-1} - p^{\alpha-2}).$$

As the remaining independent generators of $H'$ can be selected in the same number of ways from the operators of $G$ and from those of $H'$, the quotient of the given products is equal to the number of subgroups of $G$ satisfying the conditions imposed on $H'$. This number is evidently of the form $p + kp^2$, and hence the theorem in question has been proved whenever $\beta > 2$.

When $\beta = 2$ the largest independent generator of $H'$ can be selected from the operators of $G$ and from those of $H'$ in

$$p^{\alpha+\lambda} - p^{\alpha} \text{ and } p^{\alpha} - p^{\alpha-1}$$

ways respectively while the remaining independent generators of $H'$ can be selected in the same number of ways from each of these two groups. Hence the number of subgroups satisfying the conditions imposed on $H'$ is $p(p^{\lambda-1} + p^{\lambda-2} + \cdots + 1)$ in this case.

Finally, when $\beta = 1$ and $H'$ contains $\gamma > 1$ next to the largest invariants, the $\gamma$ largest independent generators of $H'$ can be selected from the operators of $G$ in

$$(p^{\alpha+1} - p^{\alpha})(p^{\delta} - p^{\delta-\gamma}) \cdots (p^{\delta} - p^{\delta-2})$$

different ways, $p^{\delta}$ being the order of the subgroup of $G$ composed of all its operators whose orders divide the second

largest invariant of $H'$. These independent generators can be selected from the operators of $H'$ in

$$(p^{\alpha} - p^{\alpha-1})(p^{\delta-1} - p^{\delta-\gamma}) \cdots (p^{\delta-1} - p^{\delta-2})$$

different ways and hence the number of these subgroups is again of the form $p + kp^2$. This completes a proof of the following theorem, since the case $\gamma = 1$ is evidently included therein:

THEOREM. *If a subgroup $H$ of an abelian group $G$ of order $p^m$ is of order $p^{\alpha}$ and composed of all the operators of $G$ whose orders divide a given number, then the number of the subgroups of $G$ whose largest invariant is $p$ times the largest invariant of $H$ and whose second invariant is obtained by dividing by $p$ the largest of the remaining invariants of $H$, while its other invariants are the same as the rest of the invariants of $H$, is always of the form $p + kp^2$.*

The preceding theorems have been established with a view to proving that the number of proper subgroups of order $p^{\alpha}$ contained in a non-cyclic abelian group $G$ of order $p^m$ is always of the form $1 + p \bmod p^2$. To complete the proof of this theorem it is only necessary to establish the fact that the number of subgroups of order $p^{\alpha}$ having a different type from those considered above must always be of the form $kp^2$, $k$ being a natural number. As a step in the proof of this theorem we note the following fundamental fact which entered the preceding considerations in a special form.

An independent generator of order $p^{\delta}$ of the subgroup $H$ of order $p^{\alpha}$ can be selected, if these generators are selected in the descending order of magnitude and $k$ of them have already been selected, from the operators of $G$ in

$$p^r - p^{s+k}$$

different ways, where $p^r$ and $p^s$ represent the orders of the subgroups of $G$ composed of all its operators whose orders divide $p^{\delta}$ and $p^{\delta-1}$ respectively. Similarly, this independent generator can be selected from the operators of $H$ in

$$p^{r'} - p^{s'+k}$$

different ways, where $r' \leqq r$ and $s' \leqq s$. To prove the theorem in question it is therefore only necessary to prove that either some $s$ is at least two units larger than the corresponding

$s'$ or at least two $s$'s are each one unit larger than the corresponding $s''$s.

From this fact it follows that the highest power of $p$ which divides the number of the subgroups of order $p^s$ and of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ may be found as follows: Determine the orders of the characteristic subgroups composed separately of all the operators whose orders divide $p^{\alpha_1-1}, p^{\alpha_2-1}, \cdots, p^{\alpha_\lambda-1}$ in $G$ and in a particular subgroup $H$ of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ respectively. The product of the orders of these subgroups of $G$ divided by the product of the orders of the corresponding subgroups of $H$ gives a quotient which is the power of $p$ in question. For instance, if $G$ is of type (6, 6, 5, 4, 2) and $H$ is of type (6, 3, 2, 1) the number of subgroups of $G$ which are of order $p^{12}$ and of type (6, 3, 2, 1) is divisible by

$$p^{21+10+5-11-7-4}$$

but not by any higher power of $p$.

In particular, it may be noted that the number of subgroups of order $p^s$ and of a given type is always divisible by $p^2$ whenever the number of the invariants of $G$ exceeds the number of the invariants of such a subgroup $H$ by 2 and at least one of the latter invariants exceeds $p$. If at least two of these invariants exceed $p$, the number of these subgroups is divisible by $p^2$ whenever $G$ has at least one more invariant than $H$ has. Hence when the number of the subgroups of the same type as $H$ has is not divisible by $p^2$ the number of invariants of $H$ is either the same as that of $G$ or one less than that of $G$ except when $H$ is of type $(1, 1, 1, \cdots)$.

Morever, when $H$ has one invariant less than $G$ and the number of subgroups having the same type as $H$ has is not divisible by $p^2$, it results from the preceding considerations that either no invariant of $H$ exceeds $p$, or that only one of these invariants exceeds $p$. In the latter case this invariant is $p^2$ unless $G$ has also only one invariant greater than $p$. Hence the following:

THEOREM: *Whenever the number of subgroups of the same type as $H$ is not divisible by $p^2$ and the number of invariants of $H$ is less than the number of invariants of $G$ there is one and only one subgroup in $G$ having the same order as $H$ but smaller invariants than $H$ has.*

It remains only to consider the case when the subgroups of order $p^s$ which have the same invariants as $H$ have as

many invariants as $G$. Suppose that the invariants of $G$ arranged in descending order of magnitude are $p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_\lambda}$ while those of $H$ arranged similarly are $p^{\alpha_1'}, p^{\alpha_2'}, \cdots, p^{\alpha_\lambda'}$. It is well known that $\alpha_1 \geqq \alpha_1', \alpha_2 \geqq \alpha_2', \cdots, \alpha_\lambda \geqq \alpha_\lambda'$. The number of the subgroups of type $(\alpha_1', \alpha_2', \cdots, \alpha_\lambda')$ is evidently divisible by $p^2$ whenever $\alpha_1'$ is at least two units larger than each of two other $\alpha'$'s which are separately smaller than the corresponding $\alpha$'s, and also when $\alpha_1'$ and $\alpha_2'$ are separately two units larger than some one $\alpha'$ which is less than the corresponding $\alpha$, or $\alpha_1'$ is at least three units larger than such an $\alpha'$ provided this $\alpha'$ is not $\alpha_2'$ and $\alpha_2 = \alpha_2' + 1$.

From the preceding paragraph it results that whenever the number of subgroups of type $(\alpha_1', \alpha_2', \cdots, \alpha_\lambda')$ is not divisible by $p^2$ and an $\alpha_\beta'$ is less than $\alpha_\beta$ then there is at most one other $\alpha'$ which is two units larger than $\alpha_\beta'$. If there is such an $\alpha'$ it is $\alpha_1'$ and $\alpha_\gamma'$, $\beta > \gamma > 1$, is equal to $\alpha_\beta' + 1$. It was noted above that when these conditions are satisfied the number of subgroups of $G$ which are of type $(\alpha_1', \alpha_2', \cdots, \alpha_\lambda')$ is of the form $p + kp^2$. This is also the case when $\alpha_\lambda' = \alpha_\lambda$, $\alpha_{\lambda-1}' = \alpha_{\lambda-1}, \ldots, \alpha_3' = \alpha_3, \alpha_2' = \alpha_2 - 1$, and $\alpha_1 < \alpha - 2$.

On the other hand, when none of the $\alpha'$'s is at least two units larger than the smallest $\alpha_\beta'$ which is less than $\alpha_\beta$ the number of subgroups of type $(\alpha_1', \alpha_2', \cdots, \alpha_\lambda')$ was proved above to be of the form $1 + p + kp^2$ provided there is at least one $\alpha'$ which exceeds $\alpha_\beta'$. If there is no such $\alpha'$ there is only one subgroup of the given type. These results establish, in particular, the following:

THEOREM. *In any non-cyclic abelian group of order $p^m$ the number of the subgroups whose order is a given proper divisor of the order of the group is always of the form $1 + p \bmod p^2$.*

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# ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY PROFESSOR TSURUICHI HAYASHI.

UNDER the same title in the November number of this BULLETIN, 1918, volume 25, page 87, Dr. Mary F. Curtis showed that, given the twisted cubic

$$x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0,$$

the condition that it be a helix is precisely the condition that it be algebraically rectifiable. Since her proof is an application of the common differential geometry, the coordinates $x_1$, $x_2$, $x_3$ are rectangular. The parametrical equations of the most general twisted cubics in rectangular coordinates $x$, $y$, $z$, are

$$x = f_1(t)/F(t), \quad y = f_2(t)/F(t), \quad z = f_3(t)/F(t),$$

where $F$, $f_1$, $f_2$, $f_3$ are polynomials of degree 3 in the parameter $t$. By increasing $t$ by a constant, and taking the axes along the tangent, the principal normal, and the binormal at the point on the curve where $t$ is infinite, these equations are reduced to

$$(1) \quad x = \frac{a_0t^2 + a_1t + a_2}{t^3 + dt + e}, \quad y = \frac{b_1t + b_2}{t^3 + dt + e}, \quad z = \frac{c_2}{t^3 + dt + e},$$

as Mr. W. H. Salmon has done in treating the twisted cubics of constant torsion.* But I will consider here the algebraic rectifiability of a less general type of twisted cubics whose equations are

$$x = a_1t^3 + a_2t^2 + a_3t + a_4,$$
$$y = b_1t^3 + b_2t^2 + b_3t + b_4,$$
$$z = c_1t^3 + c_2t^2 + c_3t + c_4.$$

This type is even more general than that treated by Dr. Curtis, and contains the cubics which have the plane at infinity as their osculating plane.

By changing the value of $t$ by a constant, transferring the origin, and taking the axes of coordinates along the tangent, the principal normal and the binormal at the point on the

* *Messenger of Mathematics*, vol. 45 (1915–1916), pp. 125–128.

curve where $t$ is then equal to zero, the equations may be written

$$(2) \quad x = a_1t^3 + a_3t, \qquad y = b_1t^3 + b_2t^2, \qquad z = c_1t^3.$$

From these equations we get

$$\Sigma\dot{x}^2 = (3a_1t^2 + a_3)^2 + (3b_1t^2 + 2b_2t)^2 + (3c_1t)^2,$$

$$\Sigma(\dot{y}\ddot{z}-\dot{z}\ddot{y})^2 = (6b_2c_1t^2)^2 + (6a_3c_1t)^2 + (-6a_1b_2t^2 + 6a_3b_1t + 2a_3b_2)^2$$

and

$$\dot{x}\ddot{y}\dddot{z} = 12a_3b_2c_1.$$

The ratio of the radii of torsion and curvature

$$\frac{T}{R} = -\left(\frac{\Sigma(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2}{\Sigma\dot{x}^2}\right)^{3/2} \cdot \frac{1}{|\dot{x}\ddot{y}\dddot{z}|}$$

can now be expressed in terms of $t$. Equating this ratio to a constant, we get only two conditions among the five coefficients $a_1$, $a_3$; $b_1$, $b_2$; $c_1$, which are necessary and sufficient in order that the twisted cubic be a helix:

$$(3) \qquad b_1 = 0 \qquad \text{and} \qquad 9a_3^2c_1^2 = 12a_1a_3b_2^2 + 4b_2^4.$$

Hence the theorem:—*In order that the twisted cubic of the type* (2) *be a helix it is necessary and sufficient that conditions* (3) *be satisfied.* The value of $T/R$ for the helix is equal to $\pm 2b_2^2/3a_3c_1$.

In this case,

$$\Sigma\dot{x}^2 = (9a_1^2 + 9c_1^2)t^4 + (6a_1a_3 + 4b_2^2)t^2 + a_3^2,$$

so that the arc of the twisted cubic

$$s = \int_{t_0}^{t_1} (\Sigma\dot{x}^2)^{1/2}dt$$

is algebraic when and only when conditions (3) hold. Hence the theorem: *The twisted cubic of the type* (2) *is algebraically rectifiable when and only when it is a helix,* a generalization of Dr. Curtis's theorem.

Lastly I will add here a theorem on twisted cubics of constant torsion and of constant curvature. It can be shown that for the twisted cubic of the type (2) of constant torsion it is necessary and sufficient that

$$(4) \qquad b_1 = 0, \qquad a_1^2 + c_1^2 = 0, \qquad 3a_3c_1^2 = 2b_1b_2^2.$$

But this is comprised in Mr. Salmon's theorem in his paper referred to. It can be also shown that for the twisted cubic of the type (2) of constant curvature, the same conditions (4) are necessary and sufficient. This was carried out by Mr. S. Narumi, one of our students. Hence the theorem: *The twisted cubics of the type* (2) *of constant torsion are of constant curvature, and conversely those of constant curvature are of constant torsion; they are all imaginary and satisfy the conditions* (4). A similar problem on the twisted cubics of the most general type (1), that is rather more interesting, as Prof. T. Kubota, one of our colleagues, says, remains unsolved.

*February*, 1919.

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# SOME GENERALIZATIONS OF THE SATELLITE THEORY.

BY PROFESSOR R. M. WINGER.

## §1. *The Rational Cubic.*

In a forthcoming paper* in the *American Journal of Mathematics* the author considers the satellite line of the cubic and some related curves. The object of the present note is to point out how the principal results there obtained can be generalized.

For the rational cubic $R^3$,

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 + 1,$$

the relation connecting a point $\tau$ and its tangential $t$ is

$$t\tau^2 + 1 = 0.$$

And the condition that $3n$ points lie on a curve $C_n$ is

$$s_{3n} = (-1)^n,$$

where $s$ refers to the product of the $t$'s†. From these equations we have at once the following theorems:

1. *The tangentials of the points in which a $C_n'$ cuts $R^3$ lie on a second $C_n$.*

* "On the satellite line of the cubic," read before the San Francisco section of the American Mathematical Society, April 6, 1918.

† Winger, "Involutions on the rational cubic," this BULLETIN, Oct., 1918.

2. *The contacts of tangents from the points in which a $C_n$ meets $R^3$ lie on a $C_{2n}'$ when n is even—and only then.*

Theorem 1 repeatedly applied yields a chain of curves of the same degree. We should naturally designate $C_n'$ and $C_n$ as primary and satellite respectively, but this terminology is hardly appropriate when $n > 2$. For then each curve is one of a pencil on the same points. We shall therefore speak of the points in question as a *primary set* $S_{3n}'$ and a *satellite set* $S_{3n}$.

Any primary set has a unique satellite set* but a satellite set has a multitude of primaries. Let a conic $C_2$ cut the cubic in six points $t_1, t_2, \cdots, t_6$. Then $s_6 = 1$. The tangentials of these points may be arranged in two sets $\tau_1, \tau_2, \cdots, \tau_6$ and $-\tau_1, -\tau_2, \cdots, -\tau_6$ such that the product of each set is unity. Call these points $A$ and $B$ and denote their products by $A_6$ and $B_6$ respectively. Selecting 6 $t$'s from points $A$ and $B$ in such a way that each subscript occurs but once, the set will lie on a conic provided it contains an even number of points $B$. There are thus 32 primary conics in all. Or, *the twelve contacts of tangents from the points of intersection of a conic and a rational cubic lie by sixes on thirty-two conics.*

In general, $n$ even, a satellite set $S_{3n}$ will have as many primary sets as there are combinations of $3n$ things $0, 2, 4, \cdots, 3n$ at a time. The total number is therefore the sum of alternate binomial coefficients, beginning with the first, of degree $3n$,† i.e., $2^{3n-1}$.

When $n$ is odd, group the points $\tau$ so that $A_{3n} = 1$, $B_{3n} = -1$. Then points $B$ but not $A$ are on a $C_n$. We shall get a primary set as before, provided we use an odd number of points $B$. The total number of such sets therefore is the sum of alternate binomial coefficients of degree $3n$ beginning with the second, again $2^{3n-1}$.

Hence in either case, *the $6n$ contacts of tangents from the intersections of $C_n$ and $R^3$ can be arranged in primary sets $S_{3n}'$ in $2^{3n-1}$ ways.*

## §2. *The General Cubic.*

Theorem 1 of the previous section can be extended at once to the general cubic by the theory of residuation. Theorem 2 is replaced by

* If theorem 1 is applied to the points of intersection of $R^3$ and $C_{2n}'$ of 2, the satellite set $S_{6n}$ is an $S_{3n}$ repeated.

† i.e., in the expansion of a binomial of degree $3n$.

3. *The $12n$ contacts of tangents from the points in which a $C_n$ meets the general cubic lie on a $C'_{4n}$.*

For if the elliptic argument $u$ is properly chosen, the intersections $u_i$ of $C_n$ satisfy the condition

$$u_1 + u_2 + \cdots + u_{3n} = \mu\omega + \mu'\omega'.*$$

The contacts of tangents from $u_i$ are

$$-\frac{u_i}{2}, \quad -\frac{u_i+\omega}{2}, \quad -\frac{u_i+\omega'}{2}, \quad -\frac{u_i+\omega+\omega'}{2},$$

the sum of which ($i = 1, \cdots, 3n$) is $-2\Sigma u_i - 3n(\omega + \omega')$, hence the theorem is proved.

Again a primary set has a unique satellite set. To enumerate the primary sets of a satellite set, consider first a cubic $C_3$ meeting the base curve in nine points $u_1, u_2, \cdots, u_9$. Denote the contacts of tangents from points $u_i$ by $A_i$, $B_i$, $C_i$ and $D_i$. A cubic $C_3'$ on eight of these points $\alpha_1, \cdots, \alpha_8$,† where the subscripts are all different, will cut again in a ninth point, say $u$. The satellite set of $C_3'$ will be cut out by a cubic on the eight points $u_1, \cdots, u_8$. But any cubic on these eight points passes through $u_9$. Hence $u_9$ is the tangential point of $u$, or $u$ is none other than a point $\alpha_9$. That is to say, any cubic on eight of the points $\alpha$, so chosen that all subscripts except one are represented, will pass through one of the four points carrying the omitted subscript and will have points $u_i$ for a satellite set. Now the points $\alpha_1$ can be chosen in four ways, $\alpha_2$ likewise and so on to $\alpha_8$, $\alpha_9$ then being determined. Hence *the points $A$, $B$, $C$, $D$ can be chosen as primary sets in $4^8$ ways.* By precisely the same reasoning, *the contacts of tangents from the $3n$ points in which a $C_n$ cuts $C_3$ can be arranged in primary sets $S_{3n}'$ in $4^{3n-1}$ ways.*

## §3. *Extension to Higher Cases.*

The satellite theory, as frequently happens, can be generalized in several directions. This is plain if it is recalled that there are three factors involved—the primary (a line), the base curve (a cubic), and the tangent lines at the points of intersection. Any one, any two or all three of these elements may be generalized. Of the more interesting cases we mention

* Pascal, Repertorium, 2d ed., II, p. 393.

† $\alpha_i$ refers to any one of the points with subscript $i$.

the following, the validity of which can be inferred from the fundamental theorem of residuation.*

4. *The n tangents at collinear points of a $C_n$ meet again in $n(n-2)$ points which lie on a $C_{n-2}$.*

5. *The mn tangents to $C_n$ at points of intersection of a $C_m$ cut $C_n$ again in points of a $C_{m(n-2)}$.*

6. *Given a base curve $C_n$, and a primary $C_m$, cutting $C_n$ in points $P_i$. Then any set of mn $C_p$'s cut $C_n$ in points of a $C_{mnp}$. If $C_p^{(i)}$ has k-point contact with $C_n$ at $P_i$ the remaining $mn(np-k)$ points constitute a satellite set.*

Theorem 6 which is itself a special case of the fundamental theorem includes all the generalizations suggested. Theorems 4 and 5 repeatedly applied yield chain theorems connecting an unending series of curves.

In particular the dual of 5, beginning with $n = 1$, establishes the chain, the first link of which was obtained by another method in the earlier paper.†

Many isolated classic theorems appear here as special cases. From 6 can be deduced readily the theorem that *if $n-1$ intersections of a line with a $C_n$ are points $P_k$ with k-point tangents, $k > 2$, the remaining intersection is a $P_k$.* This includes as a special case the familiar theorem concerning flexes.

We shall conclude with a consideration of curves $C_m$ each of which has with a base cubic a $(3m-1)$-point contact $P_{3m-1}$. Thus conics with quintactic points at intersections of a line $l'$ with a rational cubic cut again in three points of a line $l$. But from each of the latter points can be drawn five quintactic conics.‡ These 15 quintactic points lie by threes on 25 lines $l'$ five of which pass through each point.

Similarly the 24 $P_8$'s of cubics passing simply through the intersections of $l$ and $R^3$ determine a configuration of 24 points and 64 lines, 3 points on a line and 8 lines on a point.

Again if $l$ is replaced by a conic $C_2$ there will be 30 quintactic conics from points of intersection of $C_2$ and $R^3$. By reasoning analogous to that employed in §2 it is easily established that these 30 points lie by sixes on $5^5$ conics $C_2'$.

*Generally, from a point of $R^3$ can be drawn $(3m-1)$ $C_m$'s each having a contact $P_{3m-1}$ elsewhere. The 3i common points*

* Pascal, Repertorium, 2d ed., II, p. 321.
† Winger, "On the satellite line of the cubic," l.c.
‡ Winger, "Involutions on the rational cubic," l.c., p. 30.

*of a $C_i$ and $R^3$ determine thus $3i(3m-1)$ such points which can be arranged in primary sets $S'_{3i}$ of the $3i$ points in $(3m-1)^{3i-1}$ ways.*

Passing now to the general cubic, if a $C_m$ have $(3m-1)$-point contact at $u'$ and cuts again at $u$ the relation connecting the elliptic arguments is

$$u' = \frac{\mu\omega + \mu'\omega' - u}{3m-1}.$$

For given $u$ there are $(3m-1)^2$ values of $u'$ since $\mu, \mu' = 1, 2, \cdots, 3m-1$. *Thus from the common points of a conic and a $C_3$ can be drawn $6 \cdot 25$ quintactic conics whose 150 quintactic points lie by sixes on $5^{10}$ primary conics.*

*And generally from the $3i$ common points of a $C_i$ and $C_3$ can be drawn $3i(3m-1)^2$ $C_m$'s each having a $(3m-1)$-point contact. These $3i(3m-1)^2 P_{3m-1}$'s can be grouped in primary sets $S_{3i}'$ of the $3i$ points in $3(3m-1)^{6i-2}$ ways.*

UNIVERSITY OF WASHINGTON.

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## CAJORI'S HISTORY OF MATHEMATICS

*A History of Mathematics.* By FLORIAN CAJORI, Ph.D., Professor of the History of Mathematics in the University of California. New York, The Macmillan Company, 1919. vii + 514 pp. Price, $4.00.

THE present edition of Cajori's well known History of Mathematics is so completely revised and so considerably enlarged that it might almost be regarded as a new book. While it contains only about 100 more pages than the earlier edition,* it has about twice as much reading matter, the pages being larger and more closely printed than those of the first edition. The general arrangement of the subjects treated remains unchanged, but three brief new sections relating largely to ancient mathematics have been added. These are headed; "The Maya," "The Chinese" and "The Japanese" respectively, and are based on special histories relating to these peoples, which have appeared since the publication in 1894 of the first edition of the present work.

* Reviewed in this BULLETIN, vol. 3 (1894), pp. 190 and 248, by D. E. Smith and G. B. Halsted.

Naturally the most important and extensive changes appear in the sections devoted to developments made during the nineteenth century. Somewhat less than one half of the space is devoted to advances made during the nineteenth and the early part of the twentieth century. To the younger mathematical students it is likely to be very interesting and useful to find here references to the work of so many contemporary mathematicians, and to find also brief statements of the relations between contemporary developments and those due to earlier investigators.

In view of the great variety of modern mathematical advances it cannot be expected that all of them could be treated in a one-volume history. The selection of the material to be included in such a volume was a very difficult undertaking and few would be likely to agree as regards details in making such a selection. In a broad way, however, probably most mathematicians will agree that Cajori succeeded very well in selecting his material and that he treated this material with an evident effort to exhibit relative values in a true light.

To secure reliable judgments as regards the bearing and value of much of the modern work, Cajori had to depend largely on the views of specialists. He quotes these views freely and sometimes extensively. The lack of uniformity in style and conservatism thus introduced is more than compensated by the expert knowledge relating to the wide range of fields of investigation which is thus made available. As a rule this expert knowledge has been wisely utilized and references are furnished. In some cases the latter are lacking but the total number of references is large and in this respect the present edition exhibits also a great improvement on the former one.

Two important questions about a history of mathematics are: Is the work written in an attractive style? and Is it confined to established facts? In the present case our answer to the former question is yes, and to the latter, no. Our negative answer to the second of these two questions does not imply a criticism of the book, since many readers will doubtless enjoy such speculations as those in regard to the reason why the ancient Babylonians divided the circle into 360 equal parts, page 6, as well as the references to reports which are as unlikely to be true as the one relating to the sacrifice of one hundred oxen by Pythagoras, page 18.

The value of a comparatively brief general history of mathematics cannot be measured entirely by its accuracy as regards details. The field to be covered is so extensive and the number of data is so large that it is almost impossible to avoid entirely the appearance of some misleading statements. What is most important is that the main points of view are sound and that the most fundamental factors in the development of our subject are properly emphasized.

The present work appears to be satisfactory along these important lines, as might have been expected from the fact that its author has had such a broad training in writing on the history of mathematics. Many historical questions present great difficulties and call for a wide knowledge of facts as well as a deep insight into the bearing of one result on another and the tendencies, local as well as national, to claim undue credit for the work of particular investigators.

It is interesting to note that in the present edition the word *khet* is used in place of *ruth* to denote an ancient Egyptian unit of unknown length employed in the work of Ahmes. The introduction of the term ruth for this unit seems to be due to a very crude error committed by J. Gow in his History of Greek Mathematics, 1884, page 127. In making use of A. Eisenlohr's translation of the work by Ahmes, Gow apparently failed to notice that the German word Ruthen means rods, and hence he translated this word by the word ruths instead of by the word rods.

The error thus committed found its way into other histories as well as into textbooks. It appears, for instance, on page 44 of the earlier edition of the present work as well as in the revised edition of Cajori's History of Elementary Mathematics, published in 1917. Fortunately, it is corrected in the present work and it is hoped that this correction may help to banish this error from the future textbooks on elementary geometry. In making the correction Cajori wisely transliterated the ancient Egyptian word instead of attempting to give an English equivalent.

Those who are especially interested in American mathematical work will be pleased to find so many references in the volume under review to the work done by Americans. By glancing over the Index it becomes at once apparent that America is on the mathematical map,—a fact which is not always revealed by the Index of a mathematical history. On

the other hand, one fails to find in this Index as large a number of references after the name of an American as appears after the names of various foreign mathematicians. This difference does not apply only to the deceased mathematicians, although it is more pronounced in these cases. In special instances such differences may be more or less due to accident, but when wide differences of this kind present themselves as regard large collections they should not be attributed to accident and their significance becomes a matter for serious consideration.

The semi-biographical feature of the earlier edition has been retained in the present one, as well as the use of heavy type in printing the names of some of the most eminent mathematicians. The following eight names of Americans appear in such type: M. Bôcher, J. W. Gibbs, G. M. Green, G. W. Hill, E. McClintock, A. Macfarlane, S. Newcomb, and B. Peirce. While no names of living American mathematicians are printed in this way, such a restriction has not been observed in regard to European mathematicians, as the following names appear in heavy type: P. Appell, P. Bachmann, E. I. Fredholm, G. Frege, F. Klein, H. Lebesgue, K. Pearson, E. Picard, H. A. Schwarz. Several of these names could have been replaced by those of more eminent living investigators.

One of the most useful things that the reviewer of a new edition of a popular textbook can do is to direct attention to possible improvements in such a way as to be helpful to those who will use the book in the class-room. It is with this object in view that we make the following suggestions: The note on page 98 relating to our lack of knowledge as regards the origin of our common numerals appears to the reviewer to be correct, but it does not seem to be in accord with a number of statements appearing in other parts of the book, e.g., pages 55, 88, 97, and 100. It seems as if this note had been added after it was too late to revise the other statements relating to this subject. If this is the case the note could have been made much more effective by stating this fact therein even if it might appear as acknowledging that the book was in need of a revision before it was published.

On page 27 it is stated that "Menaechmus invented the conic sections," and on page 39 we read that "Menaechmus, and all his successors down to Apollonius, considered only sections of *right* cones by a plane perpendicular to their sides." The latter of these two statements seems to be especially

misleading, since both Euclid and Archimedes knew how to generate an ellipse by cutting the cone by a plane *not* perpendicular to a generator. As regards the former statement, F. Dingeldey remarked that it is probably impossible to establish the correctness of the view that Menaechmus was the discoverer of conic sections and that it is very questionable whether he knew that the curves in question can be obtained by cutting a cone by means of a plane (Repertorium der höheren Mathematik, volume 2 (1910), page 197; Encyclopédie des Sciences Mathématiques, tome 3, volume 3, page 41).

That Cajori's language is not always clear and direct may be illustrated by the following quotation from page 47: "The fundamental theorem of plane trigonometry, that two sides of a triangle are to each other as the chords of double the arcs measuring the angles opposite the two sides, was not stated explicitly by Ptolemy, but was contained implicitly in other theorems." It is difficult to see why such cumbersome language is employed here to express the well-known law of sines in regard to the plane triangle. If the object was to illustrate the language which Ptolemy might have employed to express this law, reference should have been made to this fact.

Other instances of a lack of clarity or of obvious misprints may be found as follows:

| Page. | Line. | Page. | Line. | Page. | Line. |
|---|---|---|---|---|---|
| 49 | 31 | 299 | 18 | 372 | 39 |
| 239 | 27 | 302 | 16 | 373 | 34 |
| 250 | 21 | 327 | 1 | 377 | 20 |
| " | 22 | 330 | 10 | 401 | 32 |
| 265 | 31 | 331 | 8 | 418 | 22 |
| 267 | 33 | 351 | 35 | 438 | 15 |
| 287 | 3 | 360 | 33 | 494 | 42 |
| 296 | 6 | 361 | 26 | 497 | 37 |

To test the Index as regards completeness, the places where the term "group" would be likely to appear were looked up by the reviewer and it was found that the following page numbers should be added to those following this term in the Index: 318, 323, 390, 432. On the other hand, the term group does not appear on any of the pages 362–366 and hence the series of numbers 349–366 following this term in the Index should be replaced by 349–361. The name of H. Lebesgue does not appear in the Index, although it is printed in heavy type in the text, as was noted above.

It is singular that the Index does not contain the term "fundamental theorem of algebra." This term appears on page 363 of the text and reference is made in the Index to the substance of the theorem under the term "roots." To this reference there should be added the number 253 where the following sentence appears: "In the Résolution des équations numériques (1798) he (Lagrange) gave, among other things, a proof that every equation must have a root,—a theorem which before this usually had been considered self-evident."

This statement is clearly misleading, since various demonstrations of the theorem had been attempted long before 1798. In particular d'Alembert published such a supposed demonstration in 1746 and the theorem is still frequently called D'Alembert's theorem, especially in France. Among others who had attempted to give demonstrations before 1798 is L. Euler, who furnished two such demonstrations in 1749. In fact, Lagrange had also attempted to give such demonstrations more than twenty years before 1798. The great historical importance of the fundamental theorem of algebra makes it especially desirable that the references relating thereto should be explicit and accurate.

On page 234 we read "the calculus of variations to the invention of which Euler was led by the study of the researches of Johann and Jakob Bernoulli." On the other hand, the following sentence appears on page 251. "Lagrange did quite as much as Euler towards the creation of the Calculus of Variations." If Lagrange did quite as much as Euler towards the creation of this subject it is difficult to see why Euler should be regarded as its inventor.

The lack of uniformity in the two statements noted in the preceding paragraph would be less objectionable if Euler were commonly regarded by the best modern writers as the founder of the calculus of variations. On the other hand, the vagueness involved in the term inventor of a subject and the disagreement among historical writers as to who should be regarded as the founders of various subjects tend to mitigate somewhat such apparently contradictory statements as those noted above.

There is a very noticeable lack of uniformity as regards names. That both of the names Petrograd and St. Petersburg are used for the same city is perhaps due to the somewhat recent change of its name. This is less confusing

than the adoption of spellings according to different languages of the name of the same city and the use of different initials in referring to the same person at different places. For instance, those acquainted with the works of E. Lucas will be apt to be somewhat annoyed to find his name also in the form François Edouard Anatole Lucas. It is questionable whether anything is gained by giving G. Frege's name in the form Friedrich Ludwig Gottlob Frege.

The question of form as regards names in references and in historical works presents serious difficulties in view of the lack of uniformity along this line on the part of good writers. It is frequently very undesirable to give only the family name since the number of family names represented by different mathematicians is already large and is constantly growing larger. On the other hand the names in full are often so lengthy as to become burdensome. Perhaps the most satisfactory solution is to follow the example of the large mathematical encyclopedias. At any rate, it would be desirable to have uniformity in the same work unless an explanation is made as regards the equivalent different forms used in such a work.

Since B. Peirce occupies such a prominent place in the history of American mathematics it may be of interest to quote the following from page 338: "Profound are his researches on Linear Associative Algebra. The first of several papers thereon was read at the first meeting of the American Association for the Advancement of Science in 1864." The American Association for the Advancement of Science did not hold any meeting in 1864, its first meeting was held in 1848, and its *Proceedings* for this meeting do not mention any paper on linear associative algebra.

These suggested improvements may serve to direct attention to the fact that the careful student of the history of mathematics will be inclined to make a large number of marginal notes in Cajori's book and should be encouraged to do so. On the other hand, he will find here a very useful and readable history of our subject. The numerous quotations from reliable sources, the large number of references to the work of contemporary mathematicians and the fine spirit which pervades the whole work tend to increase the value of the book. It is the largest and most modern general history of mathematics in our language and we wish for it a success commensurate with the large amount of labor involved in its preparation.

UNIVERSITY OF ILLINOIS. G. A. MILLER.

## SHORTER NOTICES.

*Mathematische Spiele.* By W. AHRENS. Third edition. [Aus Natur und Geisteswelt, No. 170.] Teubner, Leipzig and Berlin, 1916. vi + 114 pp.

THE principal new matter introduced in preparing the third edition of this booklet consists of an additional section (Magische Quadrate auf Amuletten) in the chapter on magic squares and the enlargement of the chapter on mathematical fallacies, the latter having been increased from seventeen to twenty-eight pages. To make room for this without increasing the size of the book, the whole of Chapter IV of the second edition (on Wanderungsspiele) has been omitted and also the following parts of chapters: Chapter II, §4 (Das Puzzle mit Schranken); Chapter V, §1 in part (on the number of one's ancestors); Chapter VII, §4 (Eine andere Ankleidung des Spiels Nim). Otherwise only slight changes have been made. The modifications result in increasing the interest of the booklet.

R. D. CARMICHAEL.

*Tratado Elemental de Goniometria.* Por J. DE MENDIZABAL TAMBORREL. Second edition. Mexico, Departamento de Talleres Graficos de la Secretaria de Fomento, 1917. 209 pp.

THIS textbook contains an elementary treatment of the theory and applications of the trigonometric functions. It does not correspond closely with any of the usual textbooks on this subject as we know them in this country, the difference arising primarily from the fact that the author is willing to use freely the elementary methods of the differential calculus whenever the occasion suggests it. The first of the three parts of the book is devoted to a study of the trigonometric functions; the second and third parts, to applications involving plane triangles and spherical triangles respectively. The exposition is usually suited to the needs of a learner at the stage of his development generally found among our sophomores. The book is sometimes to be criticized for lack of rigor or completeness in the proofs. A glaring instance of this sort of defect is that afforded by the derivation of the usual infinite products for sin $u$ and cos $u$ on pages 86 and 87.

R. D. CARMICHAEL.

## NOTES.

THE fourth summer meeting of the Mathematical Association of America, held at Ann Arbor in connection with the meetings of the American Mathematical Society and the American Astronomical Society, was attended by about ninety members. The meeting opened with a joint session of the three societies. The following day was devoted to discussion of the proposed mathematical dictionary and to a symposium on present day relations and tendencies between colleges and high schools as regards mathematics. The meeting closed with an exhibition of the mathematical and astronomical rarities in the library of the University of Michigan.

THE July number (volume 20, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Certain types of involutorial space transformations," by F. R. SHARPE and VIRGIL SNYDER; "On a new treatment of theorems of finiteness," by O. E. GLENN; "On the theory of developments of an abstract class in relation to the calcul fonctionnel," by E. W. CHITTENDEN and A. D. PITCHER; "On the influence of keyways on the stress distribution in cylindrical shafts," by T. H. GRONWALL; "Some convergent developments associated with irregular boundary conditions," by J. W. HOPKINS; "Groups possessing a small number of sets of conjugate operators," by G. A. MILLER.

THE List of Members of the London mathematical society, dated November 1, 1919, includes 333 names, among them those of 16 honorary members. J. E. CAMPBELL is president and T. J. I'A. BROMWICH and G. H. HARDY are secretaries of the society.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1919–1920:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mechanics of continuous media with application to elastic bodies and to perfect and viscous fluids, three hours.—By Professor L. DONATI: Theory of heat (conduction, radiation,

thermodynamics, atomistics); principle of relativity and its recurrence in several questions of modern physics, three hours.—By Professor F. ENRIQUES: Abelian integrals, three hours.—By Professor S. PINCHERLE: Theory of sets of points; functions of a real variable and their integrals; theorems of existence for ordinary differential equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Theory of elliptic functions with applications, four hours.—By Professor E. DANIELE: Mathematical theory of vibrations, four hours.—By Professor M. PICONE: Integral equations; potential; totally elliptic differential equations of mathematical physics, five hours.—By Professor G. SCORZA: Geometry on a curve from the algebraic-arithmetical standpoint of Dedekind and Weber, three hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Geometry of hyperspaces, three hours.—By Professor C. SEVERINI: Calculus of variations, four hours.—By Professor O. TEDONE: Optical phenomena of higher order: absorption and dispersion of light, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable; elliptic functions, four hours.—by Professor G. GIAMBELLI: Differential theory of singularities of plane algebraic curves (following Enriques); projective geometry of hyperspaces; theory of moduli, three hours.—By Professor O. LAZZARINO: Electrostatics, four hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics in the middle ages (from 1200 to 1600 A.D.), three hours.—By Professor A. DEL RE: $n$-dimensional extensive analysis with applications to differential geometry and mechanics, four and one half hours.—By Professor R. MARCOLONGO: Relativity, three hours.—By Professor D. MONTESANO: Rational surfaces; birational correspondences between points of ordinary space, three hours.—By Professor E. PASCAL: Monogenic functions, three hours.—By Professor L. PINTO: Geometrical optics and the theory of optical instruments, three hours.

UNIVERSITY OF PADUA. By Professor U. AMALDI: Intro-

duction to the theory of integration of Sophus Lie, four hours. —By Professor F. D'ARCAIS: Harmonic functions; functions of a complex variable; Fourier's series, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor G. RICCI: Absolute differential calculus; elasticity, four hours.—By Professor F. SEVERI: Algebraic geometry, especially of the rational surfaces, four hours.—By Professor A. TONOLO: Theorems of existence for ordinary and partial differential equations, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Calculus of variations for the functions of one independent variable; integral equations, three hours.—By Professor M. DE FRANCHIS: Geometry on algebraic curves, three hours.—By Professor M. GEBBIA: Electromagnetism with particular regard to electric oscillations, four and one half hours.—By Professor A. SIGNORINI: Hydrodynamics, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Geometry on an algebraic curve, three hours.—By Professor U. CISOTTI: Elasticity: distortions, finite strain, three hours.—By Professor F. GERBALDI: Functions of a complex variable and elliptic functions, three hours.—By Professor G. VIVANTI: Theory of algebraic equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Cremona's transformations in plane and space, three hours.—By Professor L. BIANCHI: Differential geometry, three hours.—By Professor G. A. MAGGI: Physical optics, both from the elastic and the electromagnetic standpoint, four and one half hours. —By Professor ————: Mechanics (advanced), three hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of the calculus, three hours.—By Professor E. BOMPIANI: Geometry of applicability for surfaces and varieties, three hours.—By Professor F. CANTELLI: Calculus of probabilities with applications, three hours.—By Professor G. CASTELNUOVO: Non-euclidean geometry with physical interpretations, three hours.—By Professor U. CRUDELI: Introduction to advanced theories of electricity and magnetism, three hours.—By Professor T. LEVI-CIVITA: Curves defined by differential equations; periodic solutions, three hours.—

By Professor A. PERNA: Elementary theories of mathematical analysis, three hours.—By Professor L. SILBERSTEIN: Principle of relativity, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General relativity, three hours; equations of mathematical physics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Analytical dynamics, three hours.—By Professor C. SEGRE: Groups of finite order, three hours.—By Professor C. SOMIGLIANA: Theory of electricity and magnetism, three hours.—By Professor ————: Higher analysis, three hours.

PROFESSOR S. PINCHERLE, of the University of Bologna, has been elected corresponding member of the Royal institute of Venice.

PROFESSORS G. CASTELNUOVO and T. LEVI-CIVITA, of the University of Rome, have been elected corresponding members of the Royal academy of sciences of Bologna.

PROFESSOR E. LAURA, of the University of Pavia, has been promoted to a full professorship of rational mechanics.

PROFESSOR E. BORTOLOTTI, of the University of Modena, has been appointed professor of algebra and analytic geometry at the University of Bologna.

PROFESSOR W. H. YOUNG, of the University of Liverpool, has been appointed professor of mathematics at the University College of Wales.

DR. S. CHAPMAN, chief assistant at the Greenwich Observatory, has been appointed professor of mathematics at the University of Manchester.

IN October, Professor VITO VOLTERRA, of the University of Rome, lectured at the University of California on "Propagation of electricity in a magnetic field" and "Equations involving functional derivatives." Later he will deliver a course of three lectures at the Rice Institute on "Functions of composition."

THE Royal institute of Venice has awarded the Querini-Stampalia prize to Professor G. D. BIRKHOFF, of Harvard University, for his papers: "The restricted problem of three bodies," *Rendiconti del Circolo Matematico di Palermo*, volume 39 (1915), and "Dynamical systems with two degrees of freedom," *Transactions of the American Mathematical Society*, volume 18 (1917). The subject proposed was to realize some important advance in the theory of periodic solutions of differential equations.

DR. T. M. PUTNAM, professor of mathematics and dean of the undergraduate division of the University of California, has been appointed acting dean of the college of letters and science.

DR. E. V. HUNTINGTON, associate professor of mathematics in Harvard University, has been promoted to a full professorship in mechanics. His teaching activities will be divided as heretofore between the division of mathematics and the division of engineering.

PROFESSOR J. W. GLOVER, of the University of Michigan, served as acting president of the Teachers insurance and annuity association of America during September, in the absence of President Pritchett. Professor Glover is one of the trustees of the association.

DR. T. H. GRONWALL has been appointed mathematics and dynamics expert on the technical staff of the ordnance office, U. S. war department.

PROFESSORS P. BOUTROUX and J. H. M. WEDDERBURN both return from military service to Princeton University at the opening of the present academic year.

PROFESSOR H. C. WOLFF has been appointed head of the department of mathematics at the Drexel Institute.

AT the University of Illinois, assistant professor ARNOLD EMCH has been promoted to an associate professorship of mathematics. Dr. J. R. KLINE, of Yale University, and Dr. W. W. KÜSTERMANN, of the University of Michigan, have been appointed associates, and Mr. E. H. VANCE instructor in mathematics.

PROFESSOR IDA BARNEY, of Lake Erie College, Painesville, Ohio, has been appointed head of the department of mathematics at Meredith College, Raleigh, N. C.

PROFESSOR F. W. BEAL, of the University of Tennessee, has been appointed assistant professor of mathematics at the University of Pennsylvania.

DR. E. P. LANE, of the Rice Institute, has been appointed assistant professor of mathematics in the University of Wisconsin.

PROFESSOR W. V. N. GARRETSON, of the Pennsylvania Military College, has been appointed assistant professor of mathematics at Rutgers College.

DR. M. G. SMITH has been appointed professor of mathematics at Greenville College.

In the department of mathematics of Allegheny College, Mr. K. A. MILLER has been promoted to an assistant professorship and Mr. L. M. MONROE has been appointed instructor.

MR. H. R. PHALEN has been promoted to an assistant professorship of mathematics at the Armour Institute of Technology.

MR. A. C. MADDOX has been appointed assistant professor of mathematics at the Oklahoma Agricultural and Mechanical College.

DR. T. A. PIERCE has been appointed instructor in mathematics at the University of Nebraska.

DR. E. A. T. KIRCHER has resigned from the Sheffield Scientific School to enter business.

THE following instructors in mathematics have been appointed at Cornell University: Dr. F. W. REED, of the University of Illinois, Dr. G. M. ROBISON, and Mr. H. L. VANDIVER. Mr. A. D. CAMPBELL remains at Cornell, instead of going to Yale University as stated in the July BULLETIN.

DR. EUGENE TAYLOR has been appointed assistant professor of mathematics at the University of Wisconsin. MISS MARY A. COLPITTS has been appointed instructor in mathematics.

AT Lehigh University, Mr. M. S. KNEBELMAN has been promoted to an assistant professorship of mathematics.

DR. J. V. MCKELVEY, of Cornell University, has been appointed assistant professor of mathematics in Iowa State College.

DR. FLORA E. LESTOURGEON, of Mt. Holyoke College, has been appointed assistant professor of mathematics in Carleton College, Northfield, Minn.

AT the University of Pennsylvania, DR. S. P. SHUGERT has been promoted to an assistant professorship of mathematics.

DR. TOBIAS DANTZIG, of Columbia University, has been appointed instructor in mathematics in Johns Hopkins University.

DR. L. M. KELLS, of the University of Illinois, has been appointed instructor in mathematics in the U. S. Naval Academy.

DR. S. D. ZELDIN has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

MR. H. L. SMITH, of Cornell University, has been appointed instructor in mathematics at Trinity College, Hartford.

MR. I. A. ROMAN has been appointed instructor in mathematics at Northwestern University.

PROFESSOR EDUARDO TORROJA, of the University of Madrid, died September 14, 1918, in his seventy-second year.

PROFESSOR F. E. CHAPMAN, of the Gulf Coast Military Academy, died October 1, 1919.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

CASTRO ARAUJO (P.). Analyse mathemática. Rio Janeiro, 1917.

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. 1ter Band. 4te Auflage. Leipzig, Teubner, 1918. 12 + 569 pp. Geb. M. 18.00

DAHLGREN (T.). Sur le théorème de condensation de Cauchy. (Diss.) Lund, 1918. 4to. 69 pp.

DOEHLEMANN (K.). Projektive Geometrie in synthetischer Behandlung. 4te verbesserte Auflage. Berlin, Göschen, 1918. M. 1.00

DONADT (A.). See LÜBSEN (H. B.).

HEIL (J.). Das letzte Fermatsche Theorem und seine Lösung. 2te verbesserte Auflage. Darmstadt, H. L. Schlapp, 1918.

JUNKER (F.). Höhere Analysis. 1ter Teil: Differentialrechnung. 3te verbesserte Auflage. Berlin, 1918. M. 1.00

KNOPP (K.). Funktionentheorie. 1ter Teil. 2te Auflage. (Sammlung Göschen, Nr. 668.) Leipzig, Göschen, 1918. 140 pp. M. 1.25

LÖFFLER (E.). Ziffern und Ziffernsysteme. 1ter Teil: Die Zahlzeichen der alten Kulturvölker. 2te neu bearbeitete Auflage. Leipzig, 1918. M. 1.00

LÜBSEN (H. B.). Ausführliches Lehrbuch der analytischen und höheren Geometrie. Zum Selbstunterricht. Neu bearbeitet von A. Donadt. 16te Auflage. Leipzig, 1919. M. 4.20

MILNE (J.). The analytical geometry of the straight line and the circle. (Bell's Mathematical Series.) London, Bell, 1918. 12mo. 256 pp. 5 s.

MINGOT SHELY (J.). Aritmética razonada. Granada, tip. Vazquez y Prieto, 1918. 8vo. 254 pp.

REINHARDT (K.). Ueber die Zerlegung der Ebene in Polygone. (Diss.) Frankfort a. M., 1918.

REY PASTOR (J.). Resumen de la teoria de las funciones analiticas y sus aplicaciones fisicas. Buenos Aires, 1918. 4to. 112 pp.

## II. ELEMENTARY MATHEMATICS.

BOREL (E.). Die Elemente der Mathematik. Deutsch von P. Stäckel 1ter Band: Arithmetik und Algebra. 2te Auflage. Leipzig, Teubner, 1919. 404 pp. M. 11.00

BÜRKLEN (O. T.). Formelsammlung und Repetitorium der Mathematik. 3te Auflage. Leipzig, 1918. M. 1.00

CADOVIUS (L.), FISKER (A.), HERLEV (V.) og OLSEN (L. P.). Regnebogen. Elevens Bog VII, Lærerens Bog VI. København, Gyldendalske Boghandel, 1919.

DIECK (W.). Stoffwahl und Lehrkunst im mathematischen Unterricht der Unter- und Mittelstufe höherer Lehranstalten. Leipzig, Teubner, 1918. 261 pp. M. 6.00

FISKER (A.). See CADOVIUS (I.).

FRENKEL (E.). Tabellarisch overzicht der vlakke meetkunde met gebruiksaanwijzing. Utrecht, W. Leydenroth van Boekhoven, 1917. Fl. 0.35

HART (W. W.). See WELLS (W.).

HERLEV (V.). See CADOVIUS (I.).

LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und Zahlenrechnen für Schule und Leben. Leipzig, Teubner, 1918. M. 1.40

LUNDSGAARD (E.). Regnestokken. Kortfattet Vejledning til Indøvelse i dens Brug. København, Gjellerup, 1919. 28 pp.

OLSEN (L. P.). See CADOVIUS (I.).

SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik. 1ter Teil, für die mittleren Klassen höherer Schulen. 3te Auflage. Leipzig, Teubner, 1917. 8vo. 8 + 204 pp. M. 2.60

SMITH (D. E.). See WENTWORTH (G.).

STÄCKEL (P.). See BOREL (E.).

WELLS (W.) and HART (W. W.). New high school arithmetic, academic, industrial, commercial. Chicago, Heath, 1919. 8 + 358 pp.

WENTWORTH (G.) and SMITH (D. E.). Higher arithmetic. Boston, Ginn, 1919. 6 + 250 pp. $1.00

## III. APPLIED MATHEMATICS.

ARPESANI (C.). Elementi di tecnologia meccanica: lavorazione dei legnami. 2a edizione rinnovata ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 8 + 161 pp. L. 3.50

CAMICHEL (C.), EYDOUX (D.) et GARIEL (M.). Etude théorique et expérimentale des coups de bélier. Paris, Dunod et Pinat, 1919. 4to. 398 pp.

CASTLE (F.). A manual of machine design. London, Macmillan, 1919. 9 + 351 pp. 7s. 6d.

DALE (J. H.). A course in machine drawing and sketching. London and Edinburgh, Chambers, 1919. 6 + 186 pp. 3s. 6d.

EYDOUX (D.). See CAMICHEL (C.).

FÖPPL (A.). Vorlesungen über technische Mechanik. 2ter Band: Graphische Statik. 4te Auflage. Leipzig, Teubner, 1918. 406 pp. M. 16.00

FÖRSTER (G.). See REINHERTZ (C.).

GARIEL (M.). See CAMICHEL (C.).

GIORLI (E.). L'aritmetica e la geometria dell'operaio per le scuole professionali, d'arti e mestieri, tecniche, ferroviarie ed ad uso dei capi operai ed operai. 6a edizione ampliata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 12 + 239 pp. L. 4.00

HANSTEIN (R. VON). See POSKE (F.).

HAUBER (W.). Festigkeitslehre. 5ter Neudruck. Berlin, 1918. M. 1.00

——. Statik. 2ter Teil: Angewandte Statik. 4ter Neudruck. Berlin, 1918. M. 1.00

LÖTZBEYER (P.). Grundlagen der darstellenden Geometrie mit Einschluss der Perspektive. Dresden, Ehlermann, 1918. 132 pp.

MACH (E.). Ueber Erscheinungen an fliegenden Projektilen. Vom räumlichen Sehen. Zwei Vorträge. Leipzig, Barth, 1917. 31 pp. M. 0.45

MÜLLER (E.). Lehrbuch der darstellenden Geometrie für technische Hochschulen. 1ter Band. 2te Auflage. Leipzig, Teubner, 1918. 370 pp. Geb. M. 18.00

MÜLLER (F. J.). Studien zur Geschichte der theoretischen Geodäsie. Als Vorarbeit zu einer Geschichte der Geodäsie. Augsburg, F. J. Müller, 1918. M. 6.50

NERNST (W.) und SCHOENFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 8te vermehrte und verbesserte Auflage. München, 1918. M. 17.00

POSKE (F.) und HANSTEIN (R. VON). Der naturwissenschaftliche Unterricht an den höheren Schulen. (Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht, 2te Folge, Heft 5.) Leipzig, Teubner, 1918. M. 1.40

REINHERTZ (C.). Geodäsie. 2te Auflage. Neu bearbeitet von G. Förster. Berlin, 1918. M. 1.00

SCHLICK (M.). Raum und Zeit in der gegenwärtigen Physik. Zur Einführung in das Verständnis der allgemeinen Relativitätstheorie. Berlin, Springer, 1917. 63 pp.

SCHOENFLIES (A.). See NERNST (W.).

SLATE (F.). The fundamental equations of dynamics and its main coordinate systems vectorially treated and illustrated from rigid dynamics. Berkeley, University of California Press, 1918. 7 + 233 pp.

VATER (R.). Die neueren Wärmefunktionen. 1ter Band, 5te Auflage. 2ter Band, 4te Auflage. Leipzig, Teubner, 1918. 119 + 114 pp.

VONDERHIM (J.). Schattenkonstruktionen. Neudruck. Berlin, 1918. M. 1.00

WAGNER (P.). Die Stellung der Erdkunde im Rahmen der Allgemeinbildung. (Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht, 2te Folge, Heft 6.) Leipzig, Teubner, 1918. M. 1.00